APPROXIMATE SOLUTION OF THE BOUNDARY PROBLEM FOR THE EQUATIONS OF A STATIONARY ELECTROSTATIC CHARGED-PARTICLE BEAM

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The solution is given of the equations of a three-dimensional stationary electrostatic beam of charged particles of like sign filling the region between two nearby curvilinear surfaces. We assume that the flow is nonrotational and nonrelativistic and that the velocity vector is a single-valued function. The solution is constructed in the form of an asymptotic series in powers of the small parameter ε, which is the ratio of the characteristic transverse (a) and longitudinal (l) dimensions of the problem. The first dimension is taken to be the distance between the electrodes, and l defines the scale at which the geometric and physical parameters (emitter curvature, electric field E on the emitter, and the emission current density J) change noticeably. The emission regimes limited by the space charge (ρ-regime), temperature (T-regime), and the case of nonzero initial velocity (U-regime) are studied. The asymptotic behavior is given by the formulas for the corresponding one-dimensional flow between parallel surface.

The solution of the boundary problem for emission in the ρ-regime reduces to determination of the emission current density J for fixed electrode geometry and given accelerating voltage. The corresponding formulas are presented, retaining terms of order ε^3.

Two approximations with respect to ε are performed for the T- and U-regimes. Here the unknown quantity for given properties of the emitting surface (J) will be the electric field E.

The results provided by the constructed expansions are compared with the exact solution for flow from a planar emitter along circular trajectories [1]. As an example we examine the two-dimensional problem of flow between two nearby circular cylindrical electrodes with disruption of the coaxiality.

The conventional tensor notations are used.

1. Basic equations. The regular monoenergetic nonrelativistic beam of similarly charged particles in the absence of a magnetic field in the stationary case is described by a system of differential equations, which in tensor form in the arbitrary curvilinear coordinate system q^i (i = 1, 2, 3) has the form

$$g^{ik}v_iv_k = 2\varphi + (u)^2, \quad e^{ikl}\frac{\partial v_i}{\partial q^k} = 0,$$

$$\frac{\partial}{\partial q^i}(\sqrt{g}g^{ik}\rho v_k) = 0, \quad \frac{1}{\sqrt{g}}\frac{\partial}{\partial q^i}\left(\sqrt{g}g^{ik}\frac{\partial\varphi}{\partial q^k}\right) = \rho. \tag{1.1}$$

Here v_i are the covariant velocity components, φ is the scalar potential, ρ is the space charge density, g_{ik} is the metric tensor, g is its determinant, u is a constant having the sense of the velocity at the emitter. For convenience we have omitted in (1.1) the specific charge η and 4π, which corresponds to the replacement $\eta\varphi \to \varphi$, $4\pi\eta\rho \to \rho$. With this definition $\varphi \geq 0$ and $E \geq 0$ for the ρ- and T-regimes.

We use an orthogonal system fixed to the emitter, assuming that $q^1 = 0$ is its equation in these coordinates; we define the collector by the relation $q^1_{(2)} = f(q^2, q^3)$. On the electrodes thus defined we have

$$q^1 = 0, \quad v_{q^1} = u, \quad v_{q^2} = v_{q^3} = 0, \quad \varphi = 0, \quad J_{q^1} = J(q^2, q^3), \quad E_{q^1} = E(q^2, q^3)$$
$$q^1_{(2)} = f(q^2, q^3), \quad \varphi = \varphi_{(2)}. \tag{1.2}$$

Here symbols with subscript q^1 denote the physical components of the corresponding quantities; in the ρ-regime u = E = 0 and in the T-regime u = 0.

The introduction of the characteristic scales (a in q^1, l in q^2 and q^3) makes it possible to separate ε in (1.1): The small parameter appears with the derivatives with respect to q^2 and q^3. However, it is more convenient to introduce ε

symbolically as a sign of the order of smallness of the terms following it, retaining for the notations the sense given them in (1.1). Then (1.1) is recast as follows:

$$g_{11}(v^1)^2+\varepsilon^2 g^{22}(v_2)^2+\varepsilon^2 g^{33}(v_3)^2=2\varphi+(u)^2,$$
$$\frac{\partial v_2}{\partial q^1}=\varepsilon\frac{\partial}{\partial q^2}(q_{11}v^1),\quad \frac{\partial v_3}{\partial q^1}=\varepsilon\frac{\partial}{\partial q^3}(g_{11}v^1),$$
$$\frac{\partial}{\partial q^1}(\sqrt{g}\,\rho\, v^1)+\varepsilon^2\frac{\partial}{\partial q^2}(\sqrt{g}\,\rho g^{22}v_2)+\varepsilon^2\frac{\partial}{\partial q^3}(\sqrt{g}\,\rho g^{33}v_3)=0, \tag{1.3}$$
$$\frac{\partial}{\partial q^1}\left(\sqrt{g}g^{11}\frac{\partial\varphi}{\partial q^1}\right)+\varepsilon^2\frac{\partial}{\partial q^2}\left(\sqrt{g}g^{22}\frac{\partial\varphi}{\partial q^2}\right)+\varepsilon^2\frac{\partial}{\partial q^3}\left(\sqrt{g}g^{33}\frac{\partial\varphi}{\partial q^3}\right)=\sqrt{g}\,\rho.$$

Before solving (1.3), we present the expansions for the elements of the metric tensor g_{ik} near the emitting surface, retaining terms of order ε^2:

$$g_{11}=a_0\left[1+\varepsilon\frac{a_1}{a_0^{3/2}}s+\varepsilon^2\frac{a_2}{a_0^2}s^2\right],\quad s=a_0^{1/2}q^1,$$
$$g_{22}=b_0\left[1-2\varepsilon\varkappa_1 s+\varepsilon^2\left(k_{1P}'+\varkappa_1^2-k_1^2-\delta_1\delta_2-\frac{1}{2}\frac{a_1}{a_0^{3/2}}\varkappa_1\right)s^2\right], \tag{1.4}$$
$$g_{33}=c_0\left[1-2\varepsilon\varkappa_2 s+\varepsilon^2\left(\delta_{1Q}'+\varkappa_2^2-\delta_1^2-k_1k_2-\frac{1}{2}\frac{a_1}{a_0^{3/2}}\varkappa_2\right)s^2\right].$$

Here $\varkappa_1$, $\varkappa_2$; k_1, k_2; and δ_1, δ_2 are the principal curvatures of the coordinate surfaces q^i = const, calculated for $q^1 = 0$; consequently, both the coefficients a_k of the expansion g_{11} and $\varkappa$, k, δ are functions of q^2 and q^3; P, Q are the arc lengths of the curvilinear axes q^2, q^3:

$$k_{1P}'=\frac{1}{\sqrt{g_{22}}}\frac{\partial k_1}{\partial q^2},\quad \delta_{1Q}'=\frac{1}{\sqrt{g_{33}}}\frac{\partial\delta_1}{\partial q^3}.$$

In deriving (1.4) we have considered the conditions that the space be Euclidean, formulated in terms of the principal curvatures.

2. **Solution of the beam equations.** We seek the solution of (1.3) in parametric form, defining the parameter τ in all approximations with respect to ε by the relation

$$\partial q^1/\partial\tau=v^1. \tag{2.1}$$

It can be shown that in first approximation τ coincides with the particle motion time along the trajectory.

We note the technique for integrating (1.3). From the current conservation equation accounting for (2.1) we have

$$\sqrt{g}\rho v^1=(b_0c_0)^{1/2}J-\int_0^\tau v^1\varepsilon^2\left[\frac{\partial}{\partial q^2}(\sqrt{g}\rho g^{22}v_2)+\frac{\partial}{\partial q^3}(\sqrt{g}\rho g^{33}v_3)\right]d\tau. \tag{2.2}$$

The Poisson equation makes it possible to calculate the contravariant electric field component

$$\sqrt{g}g^{11}\partial\varphi/\partial q^1=(b_0c_0)^{1/2}E+$$
$$+\int_0^\tau\left\{\sqrt{g}\rho v^1-v^1\varepsilon^2\left[\frac{\partial}{\partial q^2}\left(\sqrt{g}g^{22}\frac{\partial\varphi}{\partial q^2}\right)+\frac{\partial}{\partial q^3}\left(\sqrt{g}g^{33}\frac{\partial\varphi}{\partial q^3}\right)\right]\right\}d\tau \tag{2.3}$$

which appears in the equation of motion obtained by differentiating the energy integral with respect to q^1:

$$\frac{\partial^2 q^1}{\partial\tau^2}=g^{11}\frac{\partial\varphi}{\partial q^1}-\frac{1}{2}\varepsilon\frac{\partial\ln g_{11}}{\partial q^1}(v^1)^2-\varepsilon^2\frac{\partial}{\partial q^1}[g^{22}(v_2)^2+g^{33}(v_3)^2]. \tag{2.4}$$

Thus the problem reduces to integration of an equation of the form

$$\frac{\partial^2 q^1}{\partial\tau^2}=F(\tau;q^2,q^3).$$

where functions of preceding approximations of required order are substituted into the right-hand sides of (2.2)–(2.4). The derivatives with respect to $q^\alpha(\alpha = 2, 3)$ are calculated with the aid of the relations

$$\frac{\partial}{\partial q^\alpha}\bigg|_{q'=\text{const}} = \frac{\partial}{\partial q^\alpha}\bigg|_{\tau=\text{const}} + \frac{\partial \tau}{\partial q^\alpha}\bigg|_{q'=\text{const}} \frac{\partial}{\partial \tau}\bigg|_{q^\alpha=\text{const}}.$$

We follow this procedure and obtain the following expressions for the solution in first approximation (the symbol $\langle 1\rangle$ indicates the approximation number):

$$s\langle 1\rangle = \frac{1}{6}J\tau^3 + \frac{1}{2}E\tau^2 + u\tau + \varepsilon\Big[\frac{1}{30}J^2\Big(-\frac{5}{24}\frac{a_1}{a_0^{3/2}} + \frac{1}{6}T\Big)\tau^6 +$$
$$+ \frac{1}{20}JE\Big(-\frac{5}{6}\frac{a_1}{a_0^{3/2}} + \frac{2}{3}T\Big)\tau^5 + \frac{1}{12}E^2\Big(-\frac{3}{4}\frac{a_1}{a_0^{3/2}} + \frac{1}{2}T\Big)\tau^4 +$$
$$+ \frac{1}{12}uJ\Big(-\frac{a_1}{a_0^{3/2}} + T\Big)\tau^4 + \frac{1}{6}uE\Big(-\frac{3}{2}\frac{a_1}{a_0^{3/2}} + T\Big)\tau^3 - \frac{1}{4}u^2\frac{a_1}{a_0^{3/2}}\tau^2\Big],$$
$$2\varphi\langle 1\rangle = (1/2\,J\tau^2 + E\tau + u)^2 - u^2 + \varepsilon(1/2\,J\tau^2 + E\tau + u)\,[1/15\,J^2\tau^5 + 1/3\,JE\tau^4 +$$
$$+ 1/3\,(E^2 + 2uJ)\,\tau^3 + uE\tau^2]\,T,$$
$$b_0^{-1/2}v_2\langle 1\rangle = \varepsilon\{1/60\,J(J_P' - 5k_1J)\,\tau^5 + 1/12\,E\,(J_P' - 5k_1J)\tau^4 + 1/6[E\,(E_P' - 3k_1E) +$$
$$+ u(J_P' - 4k_1J)]\tau^3 + 1/2\,u\,(E_P' - 3k_1E)\tau^2 - u^2k_1\tau\},$$
$$c_0^{-1/2}v_3\langle 1\rangle = \varepsilon\,\{1/60\,J(J_Q' - 5\delta_1J)\tau^5 + 1/12E(J_Q' - 5\delta_1J)\tau^4 + 1/6\,[E(E_Q' - 3\delta_1E) +$$
$$+ u(J_Q' - 4\delta_1J)]\,\tau^3 + 1/2\,u(E_Q' - 3\delta_1E)\tau^2 - u^2\delta_1\tau\}. \qquad (2.5)$$

Here $T = \varkappa_1 + \varkappa_2$ is the combined curvature of the emitting surface.

The formulas defining terms of order ε^2 for $u \neq 0$ have the form

$$\Delta s\langle 2\rangle \equiv s\langle 2\rangle - s\langle 1\rangle = \sum_{k=0}^{9} A_k\tau^k + \sum_{k=0}^{3} B_k\tau^k \ln(\tau^2 + b\tau + c) +$$
$$+ \sum_{k=0}^{3} C_k\tau^k \operatorname{arctg}\frac{2\tau + b}{\sqrt{d}}, \quad b = \frac{2E}{J}, \quad c = \frac{2u}{J}, \quad d = 4c - b^2,$$
$$2\Delta\varphi\langle 2\rangle \equiv 2(\varphi\langle 2\rangle - \varphi\langle 1\rangle) = \sum_{k=0}^{10} F_k\tau^k + (J\tau^2 + 2E\tau + 2u)\Big[(B_1 + 2B_2\tau +$$
$$+ 3B_3\tau^2)\ln(\tau^2 + b\tau + c) + \Big(C_1 + 2C_2\tau + \frac{6}{\sqrt{d}}C_3\tau^2\Big)\operatorname{arctg}\frac{2\tau + b}{\sqrt{d}}\Big],$$
$$b_0^{-1/2}\Delta v_2\langle 2\rangle = \sum_{k=2}^{8}\frac{V_{k-1}}{k}\tau^k, \quad c_0^{-1/2}\Delta v_3\langle 2\rangle = \sum_{k=2}^{8}\frac{W_{k-1}}{k}\tau^k. \qquad (2.6)$$

The coefficients A, B, C, F, V, W in (2.6) depend on the geometric and physical parameters of the problem and because of their complexity they are not presented here. Let us examine in more detail the case of zero initial velocity. For $u = 0$ (2.6) is rewritten as follows:

$$\Delta s\langle 2\rangle = \sum_{k=0}^{7}\frac{D_k^\circ}{(k+1)(k+2)}\tau^{k+2} - \frac{1}{2}D_0^\circ\tau(\tau + b)\ln\Big(1 + \frac{\tau}{b}\Big),$$
$$\Delta v^1\langle 2\rangle = \sum_{k=1}^{7}\frac{D_k^\circ}{k+1}\tau^{k+1} + \frac{1}{2}D_0^\circ\Big[\tau - (2\tau + b)\ln\Big(1 + \frac{\tau}{b}\Big)\Big],$$
$$2\Delta\varphi\langle 2\rangle = \sum_{k=2}^{10}F_k^\circ\tau^k - \frac{1}{2}JD_0^\circ\tau(\tau + b)(2\tau + b)\ln\Big(1 + \frac{\tau}{b}\Big),$$
$$b_0^{-1/2}\Delta v_2\langle 2\rangle = \sum_{k=5}^{8}\frac{V_{k-1}}{k}\tau^k, \quad c_0^{-1/2}\Delta v_3\langle 2\rangle = \sum_{k=5}^{8}\frac{W_{k-1}}{k}\tau^k,$$
$$D_7 = J^3\Big[-\frac{1}{18}\frac{a_2}{a_0^2} + \frac{1}{18}\frac{a_1^2}{a_0^3} - \frac{1}{30}\frac{a_1}{a_0^{3/2}}T + \frac{1}{30}(\varkappa_1^2 + \varkappa_2^2) + \frac{7}{180}\varkappa_1\varkappa_2 - \frac{11}{126}k_1^2 -$$
$$- \frac{2}{315}\frac{J_P''}{J} - \frac{1}{1260}\frac{J_P'^2}{J^2} + \Big(\frac{11}{210}k_1 + \frac{2}{315}k_2\Big)\frac{J_P'}{J}\Big], \quad D_6 = \frac{7}{2}bD_7,$$
$$D_5 = JE^2\Big[-\frac{7}{8}\frac{a_2}{a_0^2} + \frac{7}{8}\frac{a_1^2}{a_0^3} - \frac{119}{240}\frac{a_1}{a_0^{3/2}}T + \frac{59}{120}(\varkappa_1^2 + \varkappa_2^2) + \frac{17}{30}\varkappa_1\varkappa_2 -$$
$$- \frac{161}{120}k_1^2 - \frac{1}{15}\frac{J_P''}{J} - \frac{7}{300}\frac{J_P'^2}{J^2} + \Big(\frac{119}{200}k_1 + \frac{1}{15}k_2\Big)\frac{J_P'}{J} + \frac{7}{200}\frac{J_P'}{J}\frac{E_P'}{E} -$$

$$-\frac{7}{120}\frac{E_P''}{E}-\frac{7}{120}\frac{E_P'^2}{E^2}+\left(\frac{7}{20}k_1+\frac{7}{120}k_2\right)\frac{E_P'}{E}\Bigg],$$

$$D_4=E^3\Bigg[-\frac{5}{8}\frac{a_2}{a_0^2}+\frac{5}{8}\frac{a_1^2}{a_0^3}-\frac{5}{16}\frac{a_1}{a_0^{3/2}}T+\frac{7}{24}(\varkappa_1^2+\varkappa_2^2)+\frac{1}{3}\varkappa_1\varkappa_2-\frac{7}{8}k_1^2+$$
$$+\frac{1}{20}\frac{J_P'^2}{J^2}-\frac{1}{40}k_1\frac{J_P'}{J}-\frac{19}{120}\frac{J_P'}{J}\frac{E_P'}{E}-\frac{1}{8}\frac{E_P''}{E}+\frac{1}{24}\frac{E_P'^2}{E^2}+\left(\frac{11}{12}k_1+\frac{1}{8}k_2\right)\frac{E_P'}{E}\Bigg],$$

$$D_3=\frac{E^4}{J}\left[-\frac{1}{9}\frac{J_P'^2}{J^2}+\frac{1}{18}k_1\frac{J_P'}{J}+\frac{19}{54}\frac{J_P'}{J}\frac{E_P'}{E}-\frac{5}{18}\frac{E_P'^2}{E^2}-\frac{5}{24}k_1\frac{E_P'}{E}\right],$$

$$D_2=-\frac{6}{5}bD_3,\quad D_1=\frac{9}{5}b^2D_3,\quad D_0=\frac{6}{5}b^3D_3,$$

$$F_{10}=J^4\Bigg[\frac{19}{3600}(\varkappa_1^2+\varkappa_2^2)+\frac{17}{2400}\varkappa_1\varkappa_2-\frac{1}{252}k_1^2-\frac{1}{1260}\frac{J_P''}{J}+\frac{1}{5600}\frac{J_P'^2}{J^2}+$$
$$+\left(\frac{19}{5040}k_1+\frac{1}{1260}k_2\right)\frac{J_P'}{J}\Bigg],\quad F_9=5bF_{10},$$

$$F_8=J^2E^2\Bigg[\frac{3}{16}(\varkappa_1^2+\varkappa_2^2)+\frac{1}{4}\varkappa_1\varkappa_2-\frac{89}{630}k_1^2-\frac{1}{42}\frac{J_P''}{J}+\frac{37}{25200}\frac{J_P'^2}{J^2}+$$
$$+\left(\frac{2969}{25200}k_1+\frac{1}{42}k_2\right)\frac{J_P'}{J}+\frac{41}{3600}\frac{J_P'}{J}\frac{E_P'}{E}-\frac{7}{720}\frac{E_P''}{E}-\frac{7}{720}\frac{E_P'^2}{E^2}+\left(\frac{11}{360}k_1+\frac{7}{720}k_2\right)\frac{E_P'}{E}\Bigg],$$

$$F_7=JE^3\Bigg[\frac{5}{18}(\varkappa_1^2+\varkappa_2^2)+\frac{11}{30}\varkappa_1\varkappa_2-\frac{37}{180}k_1^2-\frac{1}{45}\frac{J_P''}{J}+\frac{1}{450}\frac{J_P'^2}{J^2}+$$
$$+\left(\frac{11}{100}k_1+\frac{1}{45}k_2\right)\frac{J_P'}{J}+\frac{7}{900}\frac{J_P'}{J}\frac{E_P'}{E}-\frac{2}{45}\frac{E_P''}{E}-\frac{1}{90}\frac{E_P'^2}{E^2}+\left(\frac{29}{180}k_1+\frac{2}{45}k_2\right)\frac{E_P}{E}\Bigg],$$

$$F_6=E^4\Bigg[\frac{13}{90}(\varkappa_1^2+\varkappa_2^2)+\frac{17}{90}\varkappa_1\varkappa_2-\frac{1}{10}k_1^2-\frac{7}{900}\frac{J_P'^2}{J^2}+\frac{7}{1800}k_1\frac{J_P'}{J}+$$
$$+\frac{133}{5400}\frac{J_P'}{J}\frac{E_P'}{E}-\frac{1}{20}\frac{E_P''}{E}-\frac{1}{40}\frac{E_P'^2}{E^2}+\left(\frac{191}{1080}k_1+\frac{1}{20}k_2\right)\frac{E_P'}{E}\Bigg],$$

$$F_5=-\frac{3}{10}ED_3,\quad F_4=-\frac{10}{3}bF_5,\quad F_3=-10b^2F_5,\quad F_2=-4b^3F_5,$$

$$V_7=J^3\left[\frac{1}{90}T_P'-\frac{1}{18}k_{1S}'-\left(\frac{1}{90}\varkappa_1+\frac{1}{15}\varkappa_2\right)k_1+\frac{1}{90}T\frac{J_P'}{J}\right],\quad V_6=\frac{7}{2}bV_7,$$

$$V_5=JE^2\left[\frac{1}{120}T_P'-\frac{7}{8}k_{1S}'-\left(\frac{7}{60}\varkappa_1+\frac{119}{120}\varkappa_2\right)k_1+\frac{2}{15}T\frac{J_P'}{J}+\frac{1}{20}T\frac{E_P'}{E}\right],$$

$$V_4=E^3\left[\frac{1}{8}T_P'-\frac{5}{8}k_{1S}'-\frac{5}{8}\varkappa_2k_1+\frac{1}{6}T\frac{E_P'}{E}\right]. \qquad (2.7)$$

The symbol D° and F° means that the expressions for D and F in (2.7) must be supplemented by terms containing the derivatives of J and E with respect to Q and by the curvatures δ_1 and δ_2; these terms enter absolutely symmetrically in view of the complete equivalence of the directions q^2 and q^3 (for example, $k_1J_{P'}/J$ must be supplemented by the term $\delta_1J_{Q'}/J$); W is obtained from V by the replacement of $P \to Q$, $k \to \delta$; S is the arc length of the curvilinear axis q^1.

We note that solution (2.5) and (2.7) permits description of flows with continuous transition of the emission conditions from the T-regime to the ρ-regime.

In the case of emission limited by the space charge, the logarithmic terms in (2.7) drop out and after excluding τ expansions (2.5) and (2.7) become the series in q^1 presented in [2]. In this case, we can use the recurrence relations presented in [2] to construct the approximations with respect to ε. Retaining terms of order ε^3, we have for the potential

$$
\begin{aligned}
\frac{\varphi\langle 3\rangle}{\varphi\langle 0\rangle} = 1 &+ \varepsilon\left(\frac{1}{3}\frac{a_1}{a_0^{3/2}} + \frac{8}{15}T\right)s + \varepsilon^2\left[\frac{2}{9}\frac{a_2}{a_0^2} - \frac{1}{24}\frac{a_1^2}{a_0^3} + \frac{14}{45}\frac{a_1}{a_0^{3/2}}T + \right.\\
&\left. + \frac{83}{225}(\varkappa_1^2 + \varkappa_2^2) + \frac{157}{450}\varkappa_1\varkappa_2 - \frac{2}{9}k_1^2 - \frac{4}{45}\frac{J_P''}{J} + \frac{13}{450}\frac{J_P'^2}{J^2} + \left(\frac{1}{3}k_1 + \frac{4}{45}k_2\right)\frac{J_P'}{J}\right]s^2 + \\
&+ \varepsilon^3\left\{\frac{1}{6}\frac{a_3}{a_0^{5/2}} - \frac{7}{108}\frac{a_1 a_2}{a_0^{7/2}} + \frac{5}{324}\frac{a_1^3}{a_0^{9/2}} + \frac{28}{135}\frac{a_2}{a_0^2}T + \frac{a_1}{a_0^{3/2}}\left[\frac{83}{270}(\varkappa_1^2 + \varkappa_2^2) + \right.\right.\\
&\left. + \frac{157}{540}\varkappa_1\varkappa_2 - \frac{5}{27}k_1^2 - \frac{2}{27}\frac{J_P''}{J} + \frac{13}{540}\frac{J_P'^2}{J^2} + \left(\frac{5}{18}k_1 + \frac{2}{27}k_2\right)\frac{J_P'}{J}\right] - \frac{37}{990}T_P'' + \\
&+ \frac{112}{495}k_1\varkappa_{1P}' + \frac{19}{165}k_1\varkappa_{2P}' + \frac{37}{990}k_2 T_P' - \frac{4}{99}\varkappa_1 k_{1P}' - \frac{1}{6}k_1 k_{1S}' + \frac{31463}{111375}(\varkappa_1^3 + \varkappa_2^3) + \\
&+ \frac{199}{750}\varkappa_1\varkappa_2 T - \left(\frac{751}{2970}\varkappa_1 + \frac{28}{135}\varkappa_2\right)k_1^2 + \left(-\frac{1}{9}\varkappa_1 + \frac{5}{33}\varkappa_2\right)k_1 k_2 - \\
&- \left(\frac{988}{7425}\varkappa_1 + \frac{1018}{7425}\varkappa_2\right)\frac{J_P''}{J} + \left(\frac{689}{14850}\varkappa_1 + \frac{221}{14850}\varkappa_2\right)\frac{J_P'^2}{J^2} + \left(-\frac{233}{2475}\varkappa_{1P}' - \frac{8}{2475}\varkappa_{2P}' + \right.\\
&\left.\left. + \frac{1}{9}k_{1S}' + \frac{221}{495}\varkappa_1 k_1 + \frac{8}{45}\varkappa_2 k_1 + \frac{988}{7425}\varkappa_1 k_2 + \frac{1018}{7425}\varkappa_2 k_2\right)\frac{J_P'}{J}\right\}s^3 ,
\end{aligned}
$$

$$
2\varphi\langle 0\rangle = (^9/_2 J)^{2/3} s^{4/3}. \tag{2.8}
$$

Terms containing δ_1 and δ_2 and derivatives with respect to Q are again dropped for brevity.

3. Solution of the boundary problem. As mentioned above, for emission in the T-regime the objective is to find the field $E(q^2, q^3)$ on the emitter for given emission current density $J(q^2, q^3)$, collector geometry $s_{(2)} = a_0^{1/2} f = s_{(2)}(q^2, q^3)$ and collector potential $\varphi_{(2)}$. Thus the problem reduces to solution of the two equations

$$
\begin{gathered}
{}^1/_6 J\tau_{(2)}^3 + {}^1/_2 E\tau_{(2)}^2 + \varepsilon\Delta s_{(2)}\langle 1\rangle + \varepsilon^2\Delta s_{(2)}\langle 2\rangle = s_{(2)} , \\
[{}^1/_2 J\tau_{(2)}^2 + E\tau_{(2)}]^2 + 2\varepsilon\Delta\varphi_{(2)}\langle 1\rangle + 2\varepsilon^2\Delta\varphi_{(2)}\langle 2\rangle = 2\varphi_{(2)} .
\end{gathered} \tag{3.1}
$$

We recall the subscript (2) means that the corresponding terms are calculated on the collector. From (3.1) we have

$$
\begin{gathered}
E = V/\tau_{(2)} - {}^1/_2 J\tau_{(2)}, \quad J\tau_{(2)}^3 - 6V\tau_{(2)} + 12h = 0 , \\
V = [2\varphi_{(2)}]^{1/2}\left\{1 - \frac{1}{2}\varepsilon\frac{\Delta\varphi_{(2)}\langle 1\rangle}{\varphi_{(2)}} - \frac{1}{2}\varepsilon^2\left[\frac{1}{8}\left(\frac{\Delta\varphi_{(2)}\langle 1\rangle}{\varphi_{(2)}}\right)^2 + \frac{\Delta\varphi_{(2)}\langle 2\rangle}{\varphi_{(2)}}\right]\right\}, \\
h = s_{(2)} - \varepsilon\Delta s_{(2)}\langle 1\rangle - \varepsilon^2\Delta s_{(2)}\langle 2\rangle .
\end{gathered} \tag{3.2}
$$

In this case the discriminant of the cubic equation for $\tau_{(2)}$ is

$$
\Delta = 4J^{-3}(9Jh^2 - 2V^3) < 0 \tag{3.3}
$$

therefore there are three different real roots. The fact that $\tau_{(2)} > 0$ and $E > 0$ makes it possible to select the root satisfying the physical sense of

$$
\tau_{(2)} = 2\left(\frac{2V}{J}\right)^{1/2}\cos\frac{1}{3}(\pi + \psi), \quad \cos\psi = \frac{6h}{J}\left(\frac{J}{2V}\right)^{3/2}. \tag{3.4}
$$

The problem then reduces to calculating V and h in the first and second approximations. In so doing $\Delta s\langle 2\rangle$ and $\Delta\varphi\langle 2\rangle$ are calculated from solution (3.4) in zero approximation, while $\Delta s\langle 1\rangle$ and $\Delta\varphi\langle 1\rangle$ are calculated from first approximation.

The case of emission close to the ρ-regime ($E \to 0$) requires special analysis, since the discriminant Δ in this case approaches zero and the sign of the inequality in (3.3) is now determined by the first rather than the zero approximation. Here the solution for $\tau_{(2)}$ and E should be sought in the form of a series in powers of $\mu = \varepsilon^{1/2}$. In this case it is convenient to introduce the quantity j, which defines the deviation of the emission current density from the value given by the 3/2 law

$$
J = \frac{2}{9}\frac{[2\varphi_{(2)}]^{3/2}}{s_{(2)}^2}(1 - \mu^2 j), \quad E = \frac{2\varphi_{(2)}}{s_{(2)}}\mu\mathscr{E}, \quad \tau_{(2)} = \frac{3s_{(2)}}{[2\varphi_{(2)}]^{1/2}}(1 + \mu\vartheta). \tag{3.5}
$$

Retaining terms of order μ^3, we obtain from (2.5)

$$\frac{1}{6}J\tau_{(2)}^3+\frac{1}{2}\mu E\tau_{(2)}^2+\frac{1}{30}\mu^2J^2\left(-\frac{5}{24}\frac{a_1}{a_0^{3/2}}+\frac{1}{6}T\right)\tau_{(2)}^6+$$
$$+\frac{1}{20}\mu^3JE\left(-\frac{5}{6}\frac{a_1}{a_0^{3/2}}+\frac{2}{3}T\right)\tau_{(2)}^5=s_{(2)},$$
$$\frac{1}{4}J^2\tau_{(2)}^4+\mu JE\tau_{(2)}^3+\mu^2\left[E^2\tau_{(2)}^2+\frac{1}{30}J^3T\tau_{(2)}^7\right]+\frac{7}{30}\mu^3J^2ET\tau_{(2)}^6=2\varphi_{(2)}. \tag{3.6}$$

Substituting (3.5) into (3.6), we have

$$\mu(4\vartheta+6\mathcal{E})=-\mu^2\left[4\vartheta^2+12\vartheta\mathcal{E}-\frac{4}{3}j+\left(-\frac{1}{3}\frac{a_1}{a_0^{3/2}}+\frac{4}{15}T\right)s_{(2)}\right]-$$
$$-\mu^3\left[\frac{4}{3}\vartheta^3+6\vartheta^2\mathcal{E}-4j\vartheta+\left(-\frac{1}{3}\frac{a_1}{a_0^{3/2}}+\frac{4}{15}T\right)s_{(2)}(8\vartheta+9\mathcal{E})\right],$$
$$\mu(4\vartheta+6\mathcal{E})=-\mu^2\left[6\vartheta^2+18\vartheta\mathcal{E}+9\mathcal{E}^2-2j+\frac{4}{5}Ts_{(2)}\right]-$$
$$-\mu^3\left[4\vartheta^3+18\vartheta^2\mathcal{E}+18\vartheta\mathcal{E}^2+\left(-8j+\frac{28}{5}Ts_{(2)}\right)\vartheta+\left(-6j+\frac{42}{5}Ts_{(2)}\right)\mathcal{E}\right]. \tag{3.7}$$

Solving (3.7) for ϑ, $\mathcal{E}$,

$$\mu\mathcal{E}=\mu\mathcal{E}_0+\mu^2\left[\frac{4}{27}j+\left(\frac{1}{27}\frac{a_1}{a_0^{3/2}}-\frac{28}{135}T\right)s_{(2)}\right],$$
$$\mu\vartheta=-\frac{3}{2}\mu\mathcal{E}_0+\mu^2\left[\frac{22}{9}j-\left(\frac{7}{18}\frac{a_1}{a_0^{3/2}}+\frac{46}{45}T\right)s_{(2)}\right],$$
$$\mathcal{E}_0=\frac{1}{3}\left[\frac{4}{3}j-\left(\frac{2}{3}\frac{a_1}{a_0^{3/2}}+\frac{16}{15}T\right)s_{(2)}\right]^{1/2}. \tag{3.8}$$

Finally, let us examine the case of emission in the ρ-regime, in which the solution is given by (2.8). Solving (2.8) sequentially for J in the zero, first, and so on approximations, we obtain

$$J\langle 2\rangle=\frac{2}{9}\frac{[2\varphi_{(2)}]^{3/2}}{s_{(2)}^2}\left\{1-\varepsilon\left(\frac{1}{2}\frac{a_1}{a_0^{3/2}}+\frac{4}{5}T\right)s_{(2)}+\right.$$
$$+\varepsilon^2\left[-\frac{1}{3}\frac{a_2}{a_0^2}+\frac{13}{48}\frac{a_1^2}{a_0^3}+\frac{1}{5}\frac{a_1}{a_0^{3/2}}T-\frac{1}{50}(\varkappa_1^2+\varkappa_2^2)+\frac{163}{300}\varkappa_1\varkappa_2+\frac{1}{3}k_1^2-\right.$$
$$\left.\left.-\frac{4}{15}\frac{s''_{(2)P}}{s_{(2)}}+\frac{47}{75}\frac{s'^2_{(2)P}}{s_{(2)}^2}+\left(k_1+\frac{4}{15}k_2\right)\frac{s'_{(2)P}}{s_{(2)}}\right]s_{(2)}^2\right\}. \tag{3.9}$$

The expression for J takes a more compact form if the collector equation is specified by measuring the distance from $q^1 = 0$ along the arc length S of the curvilinear axis q^1: $S_{(2)} = S_{(2)}(q^2, q^3)$. Then we have for $J\langle 3\rangle$

$$J\langle 3\rangle=\frac{2}{9}\frac{[2\varphi_{(2)}]^{3/2}}{S_{(2)}^2}\left\{1-\frac{4}{5}\varepsilon TS_{(2)}+\varepsilon^2\left[-\frac{1}{50}(\varkappa_1^2+\varkappa_2^2)+\frac{163}{300}\varkappa_1\varkappa_2+\right.\right.$$
$$\left.+\frac{1}{3}k_1^2-\frac{4}{15}\frac{S''_{(2)P}}{S_{(2)}}+\frac{47}{75}\frac{S'^2_{(2)P}}{S_{(2)}^2}+\left(k_1+\frac{4}{15}k_2\right)\frac{S'_{(2)P}}{S_{(2)}}\right]S_{(2)}^2+$$
$$+\varepsilon^3\left[-\frac{167}{3300}T_P''+k_1\left(\frac{2}{33}\varkappa'_{1P}+\frac{5}{22}\varkappa'_{2P}\right)+\frac{167}{3300}k_2T_P'+\frac{2}{33}\varkappa_1k'_{1P}+\right.$$
$$+\frac{1}{4}k_1k'_{1S}-\frac{49}{2750}(\varkappa_1^3+\varkappa_2^3)-\frac{1}{500}\varkappa_1\varkappa_2T-\left(\frac{43}{660}\varkappa_1+\frac{2}{15}\varkappa_2\right)k_1^2+$$
$$+\left(\frac{1}{6}\varkappa_1-\frac{5}{22}\varkappa_2\right)k_1k_2-\left(\frac{124}{825}\varkappa_1+\frac{134}{825}\varkappa_2\right)\frac{S''_{(2)P}}{S_{(2)}}+$$
$$+\left(\frac{511}{1375}\varkappa_1+\frac{561}{1375}\varkappa_2\right)\frac{S'^2_{(2)P}}{S_{(2)}^2}+\left(-\frac{269}{1375}\varkappa'_{1P}+\frac{268}{1375}\varkappa'_{2P}+\frac{1}{3}k'_{1S}+\right.$$
$$\left.\left.\left.+\frac{67}{165}\varkappa_1k_1-\frac{2}{5}\varkappa_2k_1+\frac{124}{825}\varkappa_1k_2+\frac{134}{825}\varkappa_2k_2\right)\frac{S'_{(2)P}}{S_{(2)}}\right]S_{(2)}^3\right\}. \tag{3.10}$$

We see that in the first and second approximations the individual emitter segments operate independently; in this case $J\langle 1\rangle$ depends only on the over-all curvature T and the distance to the collector along q^1; with account for terms of order ε^2 a second-order differential region on the emitter is made to correspond to a region of the same order on the collecting surface. In the third approximation the current density is determined by the same region on the collector, but interaction with the neighboring regions of the emitting surface begins and this interaction can be accounted for by using the derivatives of the over-all curvature.

For $\tau_{(2)}$ we have

$$\tau_{(2)} = \frac{3s_{(2)}}{[2\varphi_{(2)}]^{1/2}}\left[1 + \varepsilon\left(\frac{1}{4}\frac{a_1}{a_0^{3/2}} + \frac{1}{5}T\right)s_{(2)}\right].$$

4. Examples. Let us examine the relationship between the results obtained by constructing the expansions in terms of ε and the exact expressions for the known analytic solution describing the flow along circular trajectories from a plane emitter [1]. We use two coordinate systems for this purpose: polar (R, ψ) and Cartesian (x, y). As the collector we take the equipotential from the exact solution

$$\psi = {}^2/_3 \arcsin\left[(2\varphi_{(2)}R^2)^{3/4}\right]. \tag{4.1}$$

and we compare the current density calculated using (3.10)

$$J\langle 2\rangle = \frac{2}{9}[2\varphi_{(2)}]^{3/2}\left[\frac{1}{S^2} + \left(-\frac{1}{25}\frac{1}{R^2} - \frac{7}{25}\frac{\psi'}{S} - \frac{4}{15}\frac{\psi''}{\psi} + \frac{47}{75}\frac{\psi'^2}{\psi^2}\right)\right], \quad S = R\psi,$$

$$J\langle 2\rangle = \frac{2}{9}[2\varphi_{(2)}]^{3/2}\left[\frac{1}{y^2} \mp \left(-\frac{4}{15}\frac{y''}{y} + \frac{47}{75}\frac{y'^2}{y^2}\right)\right]. \tag{4.2}$$

with the exact expression $J_{ex} = 1/2R^{-5}$. It is easy to see that the region in question has an upper angular bound of $60^\circ(\psi' = \infty)$ in R, ψ coordinates and the upper angular bound of $36^\circ(y' = \infty)$ in the Cartesian coordinates. Figure 1 shows the collector with potential $\varphi_2 = 0.05$ and curves of the relative error $\delta = (J_{ex} - J)/J_{ex}$ in the zero and second approximations, calculated in accordance with (4.2) (solid curves) and (4.3) (dashed curves). In R and ψ coordinates, the flow is calculated with an error less than 1% in the region $\varphi = 0$, $\varphi = 0.05$, $R = 2.27$; in x, y coordinates the corresponding region is $\varphi = 0$, $\varphi = 0.05$, $x = 1.47$. Thus, in this case conversion to the system fixed to the trajectory significantly broadens the region being considered.

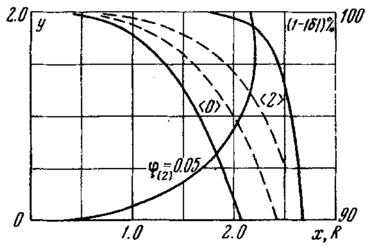

Fig. 1

We would expect that this statement is valid in the general case and can picture an iterative process of coordinate system optimization whose objective is to improve the precision of the expressions presented in sections 2 and 3.

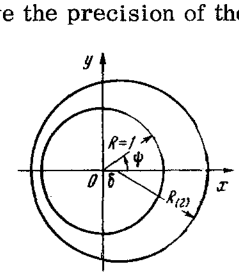

Fig. 2

As an example, consider the problem of determining the emission current density in the ρ-regime for flow between two circular cylindrical electrodes whose axes do not coincide (Fig. 2). We used a polar coordinate system fixed to the emitter ($R = 1$). Figure 3 shows the curves of $J_*\langle 3\rangle$ as a function of the polar angle ψ for two collectors ($R_{(2)} = 1.25$, solid curves; $R_2 = 1.15$, dashed curves) and several values of the distance δ between the centers. Here J_* is the current density referred to the Langmuir density for a planar diode, based on the minimum distance between the cylindrical electrodes;

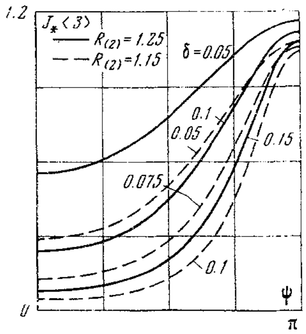
1.2
$J_*\langle 3\rangle$
$R_{(2)}=1.25$
$R_{(2)}=1.15$
$\delta=0.05$
0.1
0.05
0.15
0.075
0.1
ψ
π

Fig. 3

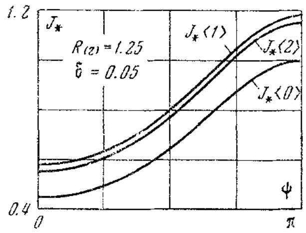
1.2
J_*
$R_{(2)}=1.25$
$\delta=0.05$
$J_*\langle 1\rangle$
$J_*\langle 2\rangle$
$J_*\langle 0\rangle$
ψ
0.4
0
π

Fig. 4

$$J_* = \frac{J}{J_L}, \quad J_L = \frac{2}{9}\frac{[2\varphi_{(2)}]^{3/2}}{[R_{(2)} - \delta - 1]^2}.$$

Figure 4 illustrates the results of the calculation of J_* in the zero, first, and second approximations for $R_{(2)} =$ $= 1.25$ and $\delta = 0.05$. The curves of $J_*\langle 2\rangle$ and $J_*\langle 3\rangle$ in the selected scale do not differ from one another (the maximum distance along the ordinate is 0.0057).

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